等質錐の基本相対不変式に付随する一般化された b-関数

Hideto Nakashima

Kyushu University

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Plan of the talk

(1) What is a b-function?
  (i) Prehomogeneous vector space
  (ii) b-function
  (iii) Properties of b-functions
(2) Symmetric cone of positive definite matrices
  (i) b-function for symmetric cone
  (ii) Known fact
(3) For general case
  (i) Setting and definitions
  (ii) matrix realization of homogeneous cones
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Prehomogeneous vector space

V : n-dim. complex vector space
G is a complex linear algebraic group
ρ: action of G on V

Definition. (G, ρ, V) is prehomogeneous vector space (PV)
$\overset{\text{def}}{\longleftrightarrow}$ G acts on V "almost" transitively by ρ.

- transitivity: for any $x, y \in V$ $\exists g \in G$ s.t. $\rho(g)x = y$.
- If G is reductive, (G, ρ, V) is also called reductive.

$\mathcal{S} = \left\{ v \in V;\ \overline{\rho(G)v} \neq V \right\}$: singular set of (G, ρ, V).

A rational function f on V is called *relative invariant* if

$\exists \chi$: a rational character s.t. $f(\rho(g)v) = \chi(g) f(v) \quad (v \in V \setminus \mathcal{S})$.

Example

Let us put $V = \mathrm{Sym}(N, \mathbb{C})$ and $G = GL(N, \mathbb{C})$.

A linear map ρ is defined by

$$\rho(g)x := gx\,{}^t g \quad (g \in G,\ x \in V).$$

Then G acts on V almost transitively by ρ.

The singular set is given by

$$\mathcal{S} = \{x \in V;\ \det x = 0\}.$$

$f(x) = \det x$ is a relatively invariant polynomial.

Dual prehomogeneous vector space

$V^* = \{f : V \to \mathbb{C};$ linear map$\}$: dual vector space of V
$\to$ write $f(v) = \langle v, f \rangle$

contragredient representation ρ^* of ρ is defined through

$$\langle \rho(g)v, \rho^*(g)f \rangle = \langle v, f \rangle \quad (g \in G,\ f \in V^*,\ v \in V).$$

The triplet (G, ρ^*, V^*) is not PV in general, but if so, we call it the dual PV of (G, ρ, V).

Theorem (cf. Kimura)

Let (G, ρ, V) be reductive.

1. (G, ρ^*, V^*) is also PV.
2. $f(x)$: relatively inv. poly. of degree $d \longleftrightarrow \chi$
   $\Rightarrow \exists f^*(y)$: relatively inv. poly. of degree d of the dual PV $\longleftrightarrow \chi^{-1}$.

b-function

Assume that G is reductive.
For a polynomial $p(x)$, a differential operator $p(D_x)$ is given as

$$p(D_x)e^{\langle x \mid y \rangle} = p(y)e^{\langle x \mid y \rangle}.$$

$(\langle x \mid y \rangle = x_1 y_1 + \cdots + x_r y_r)$

Definition. b-function of f is defined through

$$f^*(D_x)f(x)^{s+1} = b(s)f(x)^s.$$

Properties of b-function:

1. Any b-function is of the form $b(s) = b_0 \prod_j (s + \alpha_j)$ with positive rational numbers α_j.
2. $b(s)$ satisfies the following functional equation:
$$b(s) = (-1)^d b\left(-s - \frac{n}{d} - 1\right).$$
3. b-functions control the poles of zeta functions of PV.

Relationship between homogeneous cones and PV

V: finite dimensional real vector space
Ω: homogeneous cone in V

$$\begin{array}{ccccl} V & \supset \Omega & \overset{\rho}{\curvearrowleft} & H & \text{: split solvable Lie group} \\ \downarrow & & & \downarrow & \text{complexification} \\ W & & \overset{\rho_{\mathbb{C}}}{\curvearrowleft} & H_{\mathbb{C}} & \text{: almost transitive} \end{array}$$

$\to (H_{\mathbb{C}}, \rho_{\mathbb{C}}, W)$: solvable preghomogeneous vector space
In particular, not reductive in general.
Since the b-function does not depend on an $\mathbb{R}$-structure, we consider H.

$$\begin{array}{cccc} & (G,\rho,V) & \longleftrightarrow & (G,\rho^*,V^*) \\ \text{PV} & | & & | \\ & f & & f^* \\ \hline & (H,\rho,\Omega) & \longleftrightarrow & (H,\rho^*,\Omega^*) \\ \text{cone} & | & & | \\ & \Delta_{\underline{s}} & & \Delta^*_{\underline{s}^*} \end{array}$$

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Symmetric cone of positive definite matrices

$\mathcal{S}_r$: symmetric matrices
$\mathcal{S}_r^+$: positive definite matrices in $\mathcal{S}_r$
$\mathcal{H}_r$: lower triangular matrices with positive diagonals
$\to \mathcal{H}_r$ acts simply transitively on $\mathcal{S}_r^+$ by

$$\rho(h)x := hx\,{}^t h \quad \text{(Colesky decomposition)}\,.$$

The dual cone $(\mathcal{S}_r^+)^* = \mathcal{S}_r^+$.
Identify $\mathcal{S}_r^*$ with $\mathcal{S}_r$ through $\langle\, x \,|\, y \,\rangle := \mathrm{tr}\, xy$.

$\rho^*(h)y := {}^t(h^{-1})\, y\, (h^{-1})$: contragredient representation.
Then $\mathcal{H}_r$ acts simply transitively by ρ^* on $(\mathcal{S}_r^+)^* = \mathcal{S}_r^+$.

For $\underline{\nu} = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$, $\chi_{\underline{\nu}}$ below is a rational character of $\mathcal{H}_r$:

$$\chi_{\underline{\nu}}(h) = h_1^{2\nu_1} \cdots h_r^{2\nu_r} \quad (h \in \mathcal{H}_r).$$

$h_1, \ldots, h_r$: diagonal elements of h.

Symmetric cones

For $j = 1, \ldots, r$, put

$$\begin{array}{lll} \Delta_j(x) := \det^{(j)}(x) & : & \text{left upper principal minors} \\ \Delta_j^*(y) := \det_{[r-j+1]}(y) & : & \text{right lower corner principal minors} \end{array}$$

Then one has

$$\Delta_k(\rho(h)x) = h_1^2 \cdots h_k^2 \Delta_k(x), \ \Delta_k^*(\rho^*(h)y) = h_k^{-2} \cdots h_r^{-2} \Delta_k^*(y),$$

that is, $\Delta_j(x)$ (resp. $\Delta_j^*(y)$) is a relatively inv. poly. of $\mathcal{S}_r$ (resp. $\mathcal{S}_r^*$). Define

$$\begin{cases} \Delta_{\underline{s}}(x) := \Delta_1(x)^{s_1 - s_2} \cdots \Delta_{r-1}(x)^{s_{r-1} - s_r} \Delta_r(x)^{s_r}, \\ \Delta_{\underline{s}^*}^*(y) := \Delta_1^*(y)^{s_1^*} \Delta_2^*(y)^{s_2^* - s_1^*} \cdots \Delta_r^*(y)^{s_r^* - s_{r-1}^*}. \end{cases}$$

Then

$$\Delta_{\underline{s}}(\rho(h)x) = \chi_{\underline{s}}(h) \Delta_{\underline{s}}(x), \quad \Delta_{\underline{s}^*}^*(\rho^*(h)y) = \chi_{\underline{s}^*}^{-1}(h) \Delta_{\underline{s}^*}^*(y).$$

Symmetric cones

For $f(x) = \Delta_{\underline{s}}(x)$, we put $f^*(y) = \Delta^*_{\underline{s}}(y)$.

Assume $f^*(y)$ is polynomial. Then we have

$$s_r \geq s_{r-1} \geq \cdots \geq s_1 \geq 0.$$

Now we can consider the following:

$$f^*(D_x)f(x)^{s+1} = b(s)f(x)^s.$$

Remark

$f(x)$ is not polynomial in general. Let $r = 2$. Then we have

$$\Delta_1(x) = x_{11}, \quad \Delta_2(x) = x_{11}x_{22} - x_{21}^2,$$
$$\Delta_1^*(y) = y_{11}y_{22} - y_{21}^2, \quad \Delta_2^*(y) = y_{22}.$$

If $s_2 > s_1$, then

$$f(x) = \frac{(x_{11}x_{22} - x_{21}^2)^{s_2}}{x_{11}^{s_2 - s_1}}, \quad f^*(y) = (y_{11}y_{22} - y_{21}^2)^{s_1} y_{22}^{s_2 - s_1}.$$

Known result

Theorem (cf. Faraut–Koranyi)

If $\underline{s} = (s_1, \ldots, s_r)$ satisfies $s_r \geq s_{r-1} \geq \cdots \geq s_1 \geq 0$, then b-function of $\Delta_{\underline{s}}(x)$ is calculated as

$$b(s) = \prod_{i=1}^{r} \prod_{j=0}^{s_i - 1} \left(s_i s + 1 + \frac{r-i}{2} + j \right).$$

To prove this theorem, they used Laplace transforms of symmetric cones. This method can be generalized to any homogeneous cones.

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Homogeneous cones

V : finite dimensional real vector space with $\langle \cdot | \cdot \rangle$
$\Omega \subset V$: open convex cone containing no entire line
$G(\Omega) = \{g \in GL(V);\ g(\Omega) = \Omega\}$: Lie group
$\Omega^* = \{y \in V;\ \langle x | y \rangle > 0\ \forall x \in \overline{\Omega} \setminus \{0\}\}$: dual cone

Definition.

(1) Ω is homogeneous $\overset{\text{def}}{\longleftrightarrow}$ $G(\Omega)$ acts on Ω transitively
(2) Ω is self adjoint $\overset{\text{def}}{\longleftrightarrow}$ $\Omega = \Omega^*$ (with suitable inner prod.)
(3) Ω is symmetric cone $\overset{\text{def}}{\longleftrightarrow}$ homogeneous and self adjoint

Example.

(1) $\Omega = \mathrm{Sym}(r, \mathbb{R})^+,\ \mathrm{Herm}(r, \mathbb{C})^+$, Lorentz cone.
These are symmetric cones.

(2) None-symmetric cone (Vinberg cone):

$$\Omega = \left\{ \left(\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}, \begin{pmatrix} x_1 & x_4 \\ x_4 & x_5 \end{pmatrix} \right);\ x_1 > 0,\ \begin{matrix} x_1x_3 - x_2^2 > 0 \\ x_1x_5 - x_4^2 > 0 \end{matrix} \right\}.$$

Matrix realization of homogeneous cones (Ishi 2006)

$N = n_1 + \cdots + n_r$: partition of $N \in \mathbb{N}$

$\mathcal{V}_{lk} \subset \mathrm{Mat}(n_l, n_k; \mathbb{R})$: system of vector spaces satisfying

(V0) $\mathcal{V}_{jj} = \mathbb{R} I_{n_j}$ $(j = 1, \ldots, r)$,

(V1) $A \in \mathcal{V}_{lk}$, $B \in \mathcal{V}_{kj} \Rightarrow AB \in \mathcal{V}_{lj}$ $\quad (j < k < l)$,

(V2) $A \in \mathcal{V}_{lj}$, $B \in \mathcal{V}_{kj} \Rightarrow A\,{}^tB \in \mathcal{V}_{lk}$ $\quad (j < k < l)$,

(V3) $A \in \mathcal{V}_{kj} \Rightarrow A\,{}^tA \in \mathcal{V}_{kk} = \mathbb{R} I_{n_k}$ $\quad (j < k)$.

$$\mathcal{Z}_{\mathcal{V}} = \left\{ X = \begin{pmatrix} x_1 I_{n_1} & {}^tX_{21} & \cdots & {}^tX_{r1} \\ X_{21} & x_2 I_{n_2} & \ddots & {}^tX_{r2} \\ \vdots & & \ddots & \\ X_{r1} & X_{r2} & \cdots & x_r I_{n_r} \end{pmatrix};\ \begin{matrix} x_{kk} \in \mathbb{R} \\ X_{lk} \in \mathcal{V}_{lk} \end{matrix} \right\} \subset \mathcal{S}_N,$$

$\mathcal{P}_{\mathcal{V}} = \{X \in \mathcal{Z}_{\mathcal{V}};\ X \text{ is positive definite}\}$.

$\rightarrow \mathcal{P}_{\mathcal{V}}$ is a homogeneous cone of rank r.

Any homogeneous cone Ω can be realized as some $\mathcal{P}_{\mathcal{V}}$.

Split solvable Lie subgroup H

$\exists H$: split solvable Lie subgr. of $G(\Omega)$ s.t. $H \curvearrowright \Omega$ simply transitively.

Example

Let $\Omega = \mathcal{S}_r^+$.
Then $H = \mathcal{H}_r$: lower triangular matrices with positive diagonals.
In fact, $\mathcal{H}_r$ acts simply transitively on Ω by $h \cdot x = hx\,{}^th$.

H is linearly isomorphic to

$$\left\{ h = \begin{pmatrix} h_1 I_{n_1} & & & \\ T_{21} & h_2 I_{n_2} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & \cdots & h_r I_{n_r} \end{pmatrix} ;\ \begin{array}{l} h_k > 0 \\ T_{lk} \in \mathcal{V}_{lk} \end{array} \right\} \subset \mathcal{H}_N.$$

The action on $\mathcal{P}_{\mathcal{V}}$ is described as $h \cdot x = hx\,{}^th$.

Relatively H-invariant functions

Definition. f: relatively H-invariant function of Ω

$$\overset{\text{def}}{\Longleftrightarrow} \exists\chi\colon H \to \mathbb{R}\text{: 1-dim. rep. s.t. } f(h\cdot x) = \chi(h)f(x).$$

$\to \exists\underline{\nu} = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$ (multiplier) s.t.

$$\chi(h) = h_1^{2\nu_1} \cdots h_r^{2\nu_r}.$$

Put $\quad \mathrm{diag}(x_1, \ldots, x_r) = \begin{pmatrix} x_1 I_{n_1} & & \\ & \ddots & \\ & & x_r I_{n_r} \end{pmatrix} \in \Omega.$

Then $x_j > 0$ and we have

$$f(\mathrm{diag}(x_1, \ldots, x_r)) = x_1^{\nu_1} \cdots x_r^{\nu_r}.$$

Basic relative invariants

Theorem (Ishi–Nomura 2008)

There exist just r relatively H-invariant irred. polys $\Delta_1(x), \ldots, \Delta_r(x)$, and any relatively H-invariant polynomial $p(x)$ is written as

$$p(x) = (\text{const.})\Delta_1(x)^{m_1} \cdots \Delta_r(x)^{m_r} \quad (m_j \in \mathbb{Z}_{\geq 0}).$$

Moreover Ω is described as $\Omega = \{x \in V;\ \Delta_j(x) > 0 \text{ for all } j\}$.

$\to \Delta_1, \ldots, \Delta_r$ are called the basic relative invariants of Ω.

$\underline{\sigma}_j = (\sigma_{j1}, \ldots, \sigma_{jr})$: multiplier of $\Delta_j(x)$

$$\sigma := \begin{pmatrix} \underline{\sigma}_1 \\ \vdots \\ \underline{\sigma}_r \end{pmatrix} = (\sigma_{jk})_{1 \leq j,k \leq r}: \text{ multiplier matrix}$$

multiplier matrix is lower, the diagonal elements are all 1 (Ishi 2001).
We have an algorithm for calculating σ (N-. 2014).

Dual cones

$\Omega^* = \{y \in V;\ \langle x \,|\, y \rangle > 0\ \forall x \in \overline{\Omega} \setminus \{0\}\}$.

Put $\rho(h)x = h \cdot x = hx\,{}^t\!h$.
Let ρ^* be the linear map defined through

$$\langle \rho^*(h)x \,|\, \rho(h)y \rangle = \langle x \,|\, y \rangle .$$

$\to$ ρ^* is described as $\rho^*(h)y = {}^t(h^{-1})\, y\, (h^{-1})$ $(y \in V)$.
Then H acts simply transitively on Ω^* by ρ^*.

Thus Ω^* is also a homogeneous cone.

Definition. f^* is an H-relatively invariant function of Ω^*
$\overset{\text{def}}{\longleftrightarrow}$ $\exists \chi \colon H \to \mathbb{R}$: 1-dim. rep. s.t. $f^*(\rho^*(h)y) = \chi^{-1}(h) f^*(y)$.

If $\chi = \chi_{\underline{\nu}}$, then $\underline{\nu}$ is called the multiplier of f^*.

σ_*: multiplier matrix of Ω^*, which is upper triangular.

Example 1

Put $\boldsymbol{V} = \boldsymbol{\mathcal{S}_r}$.

- Symmetric cone of positive definite matrices:

$$\boldsymbol{\Omega} = \{\boldsymbol{x} \in \boldsymbol{V};\ \boldsymbol{x} \text{ is positive definite}\}.$$

- Basic relative invariants: $\boldsymbol{\Delta_j(x) = \det^{(j)}(x) \quad (j = 1, \ldots, r)}$.
- Dual cone: $\boldsymbol{\Omega^* = \{y \in V;\ y}$ is positive definite$\boldsymbol{\} (= \Omega)}$.
- Basic relative invariants of $\boldsymbol{\Omega^*}$:

$$\boldsymbol{\Delta_k^*(y) = \det_{[k]}(y) \quad (k = 1, \ldots, r).}$$

- Multiplier matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma_*}$:

$$\boldsymbol{\sigma} = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \quad \boldsymbol{\sigma_*} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}.$$

Example 2

Put $$V = \left\{ x = \begin{pmatrix} x_1 & 0 & x_2 & 0 \\ 0 & x_1 & 0 & x_4 \\ x_2 & 0 & x_3 & 0 \\ 0 & x_4 & 0 & x_5 \end{pmatrix} ; \ x_1, \dots, x_5 \in \mathbb{R} \right\}.$$

- Vinberg cone: $\Omega = V \cap \mathcal{S}_4^+$.
  This is the least dimensional non-symmetric homogeneous cone.
- Basic relative invariants:

$$\Delta_1(x) = x_1, \quad \Delta_2(x) = x_1 x_3 - x_2^2, \quad \Delta_3(x) = x_1 x_5 - x_4^2.$$

- Multiplier matrix $\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Example 2

Put $V^* = \left\{ y = \begin{pmatrix} y_1 & y_2 & y_4 \\ y_2 & y_3 & 0 \\ y_4 & 0 & y_5 \end{pmatrix} ;\ y_1, \dots, y_5 \in \mathbb{R} \right\}$.

- dual coupling: for $x \in V$ and $y \in V^*$

$$\langle x, y \rangle := x_1 y_1 + 2x_2 y_2 + x_3 y_3 + 2x_4 y_4 + x_5 y_5.$$

- dual Vinberg cone: $\Omega^* := V^* \cap \mathcal{S}_3^+$.
- Basic relative invariants of Ω^*:

$$\Delta_1^*(y) = \det y, \quad \Delta_2^*(y) = y_3, \quad \Delta_3^*(y) = y_5.$$

- Multiplier matrix $\sigma_* = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Generalized b-functions

In the case of prehomogeneous vector spaces:

$$f^*(D_x)f(x)^{s+1} = b(s)f(x)^s$$

where f^* and f have the "same" multiplier.

In the case of homogeneous cones

$$\begin{array}{llcc} & \text{relative invariant} & & \text{multiplier} \\ \Omega: & f(x) = \Delta_1(x)^{\nu_1}\cdots\Delta_r(x)^{\nu_r} & \longleftrightarrow & \underline{\nu}\sigma \\ \Omega^*: & f^*(y) = \Delta_1^*(y)^{\nu_1'}\cdots\Delta_r^*(y)^{\nu_r'} & \longleftrightarrow & \underline{\nu}'\sigma_* \end{array}$$

If $\underline{\nu}' = \underline{\nu}\sigma\sigma_*^{-1}$, then f and f^* have the same multiplier.

In order to consider the analogous, $f^*(y)$ needs to be polynomial.

Generalized b-functions

Lemma 1.

$f^*(y)$ is polynomial if and only if $\underline{\nu}' = \underline{\nu}\sigma\sigma_*^{-1} \in \mathbb{Z}^r_{\geq 0}$.

$\rightarrow$ Assume that $\underline{\nu}' \in \mathbb{Z}^r_{\geq 0}$.

Proposition (cf. Kimura)

Let $f(x) = \Delta^{\underline{\nu}}(x)$ with $\underline{\nu}' \in \mathbb{Z}^r_{\geq 0}$. Then there exists a function $b(s)$ such that

$$f^*(D_x)f(x)^{s+1} = b(s)f(x)^s.$$

$\Rightarrow$ $b(s)$: generalized b-function of relatively H-invariant func. f of Ω.

Gamma function

Gamma function $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx$ $(s > 0)$.

$\to$ generalize to homogeneous cones.

We consider the following integral:

$\Delta^*_{\underline{s}^*}(y)$: relatively H-inv. func. whose multiplier is $\underline{s}^* \in \mathbb{R}^r$,

$d\mu(y)$: H-invariant measure,

$$\Gamma_{\Omega^*}(\underline{s}^*) := \int_{\Omega^*} e^{-\mathrm{tr}(y)} \Delta^*_{\underline{s}^*}(y)\, d\mu(y).$$

Theorem (Gindikin)

The integral converges if and only if $s_j > m_j/2$.

If converges, then we have

$$\Gamma_{\Omega^*}(\underline{s}^*) = \frac{\pi^{(\dim V - r)/2}}{2^r} \prod_{j=1}^{r} \Gamma\left(s^*_j - \frac{m_j}{2}\right),$$

where $m_j := \sum_{k>j} \dim \mathcal{V}_{kj}$.

Laplace transform

$\Delta_{\underline{s}}(x)$: relatively H-inv. func. of Ω whose multiplier is $\underline{s} \in \mathbb{R}^r$.

Theorem (Gindikin)

The integral $\mathcal{L}[\Delta^*_{\underline{s}^*}](x) := \dfrac{1}{\Gamma_{\Omega^*}(\underline{s}^*)} \displaystyle\int_{\Omega^*} e^{-\langle x | y \rangle} \Delta^*_{\underline{s}^*}(y)\, d\mu(y)$

converges if and only if $s_j^* > m_j/2$, and if converges one has

$$\mathcal{L}[\Delta^*_{\underline{s}^*}](x) = \Delta_{-\underline{s}^*}(x).$$

For $x \in \Omega$ and $y \in \Omega^*$, we put

$$\Delta^{\underline{\nu}}(x) := \Delta_1(x)^{\nu_1} \cdots \Delta_r(x)^{\nu_r}, \quad \Delta_*^{\underline{\nu}^*}(y) := \Delta_1^*(y)^{\nu_1^*} \cdots \Delta_r^*(y)^{\nu_r^*}.$$

Then we have $\Delta^{\underline{\nu}}(x) = \Delta_{\underline{\nu}\sigma}(x)$ and $\Delta_*^{\underline{\nu}^*}(y) = \Delta^*_{\underline{\nu}^*\sigma_*}(y)$, and hence

$$\mathcal{L}[\Delta_*^{\underline{\nu}^*}](x) = \frac{1}{\Delta^{\underline{\widetilde{\nu}}}(x)} \quad (x \in \Omega).$$

Example

$\boldsymbol{\Omega}$: Vinberg cone

$$\Delta_1(x) = x_1, \quad \Delta_2(x) = x_1x_3 - x_2^2, \quad \Delta_3(x) = x_1x_5 - x_4^2,$$
$$\Delta_1^*(y) = \det y, \quad \Delta_2^*(y) = y_3, \quad \Delta_3^*(y) = y_5.$$

In this case, $\sigma_*\sigma^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

Put $\underline{\nu}^* = (\nu_1^*, \nu_2^*, \nu_3^*)$. Then we have

$$\widetilde{\underline{\nu}^*} = (\nu_1^*, \nu_2^*, \nu_3^*)\sigma_*\sigma^{-1} = (-\nu_1^* - \nu_2^* - \nu_3^*, \nu_1^* + \nu_2^*, \nu_1^* + \nu_3^*),$$

and hence

$$\Delta_1^*(y)^{\nu_1^*}\Delta_2^*(y)^{\nu_2^*}\Delta_3^*(y)^{\nu_3^*} \xrightarrow{\mathcal{L}} \frac{\Delta_1(x)^{\nu_1^*+\nu_2^*+\nu_3^*}}{\Delta_2(x)^{\nu_1^*+\nu_2^*}\Delta_3(x)^{\nu_1^*+\nu_3^*}}.$$

Main theorem

Theorem

Let $f(x) = \Delta^{\underline{\nu}}(x)$ with $\underline{\nu}' \in \mathbb{Z}^r_{\geq 0}$. Put $\underline{\gamma} = \underline{\nu}\sigma$. Then $b(s)$ is given as

$$b(s) = \prod_{j=1}^{r} \prod_{k=0}^{\gamma_j - 1} (\gamma_j s + \widetilde{m}_j + k),$$

where $\widetilde{m}_j = 1 + (\sum_{k>j} \dim \mathcal{V}_{kj})/2$. (cf. Gindikin 1992)

- If we write $b(s) = b_0 \prod_{j,k}(s + \alpha_{jk})$, then

$$b_0 = \prod_j (\gamma_j)^{\gamma_j}, \quad \alpha_{jk} = \frac{\widetilde{m}_j + k}{\gamma_j} \quad (\gamma_j \neq 0).$$

  In particular, each α_{jk} is a positive rational number.
  (This is shown in more general scheme by Kashiwara '76)

Sketch of the proof

Put $\underline{\gamma} = \underline{\nu}\boldsymbol{\sigma}$ and $|\underline{\gamma}| = \gamma_1 + \cdots + \gamma_r$.

Assume that $s\gamma_j < -m_j/2$ for all j.

Since $f(x)^{s+1} = \Delta^{(s+1)\underline{\nu}}(x) = \mathcal{L}[\Delta^{-(s+1)\underline{\nu}'}](x)$, we have

$$\begin{aligned}
f^*(D_x)f(x)^{s+1} &= \frac{\Delta_*^{\underline{\nu}'}(D_x)}{\Gamma_{\Omega^*}(-(s+1)\underline{\gamma})}\int_{\Omega^*} e^{-\langle x\,|\,y\rangle}\Delta_*^{-(s+1)\underline{\nu}'}(y)\,d\mu(y) \\
&= \frac{(-1)^{|\underline{\gamma}|}}{\Gamma_{\Omega^*}(-(s+1)\underline{\gamma})}\int_{\Omega^*} e^{-\langle x\,|\,y\rangle}\Delta_*^{-s\underline{\nu}'}(y)\,d\mu(y) \\
&= (-1)^{|\underline{\gamma}|}\frac{\Gamma_{\Omega^*}(-s\underline{\gamma})}{\Gamma_{\Omega^*}(-(s+1)\underline{\gamma})}\Delta^{s\underline{\nu}}(x).
\end{aligned}$$

Thus, we obtain

$$b(s) = (-1)^{|\underline{\gamma}|}\frac{\Gamma_{\Omega^*}(-s\underline{\gamma})}{\Gamma_{\Omega^*}(-(s+1)\underline{\gamma})}.$$

Sketch of the proof

C = constant of gamma function, and $\widetilde{m}_j := 1 + m_j/2$.

$$\Gamma_{\Omega^*}(-s\underline{\gamma}) = C \prod_{j=1}^{r} \Gamma\left(-s\gamma_j - \frac{m_j}{2}\right)$$

$$= C \prod_{j=1}^{r} \left(-s\gamma_j - \frac{m_j}{2} - 1\right) \cdots \left(s\gamma_j - \frac{m_j}{2} - \gamma_j\right)$$

$$\times \Gamma\left(-s\gamma_j - \frac{m_j}{2} - \gamma_j\right)$$

$$= \prod_{j=1}^{r} \prod_{k=0}^{\gamma_j - 1} (-s\gamma_j - \widetilde{m}_j - k) \times C \prod_{j=1}^{r} \Gamma\left(-(s+1)\gamma_j - \frac{m_j}{2}\right)$$

$$= (-1)^{|\underline{\gamma}|} \prod_{j=1}^{r} \prod_{k=0}^{\gamma_j - 1} (s\gamma_j + \widetilde{m}_j + k) \times \Gamma_{\Omega^*}(-(s+1)\underline{\gamma}).$$

Thus we obtain $b(s) = \prod_{j=1}^{r} \prod_{k=0}^{\gamma_j - 1} (\gamma_j s + \widetilde{m}_j + k)$.

Additional fact

Ω is irreducible $\Leftrightarrow$ $\Omega = \Omega_1 \oplus \Omega_2$ implies $\Omega_1 = \{0\}$ or $\Omega_2 = \{0\}$.

Theorem (N.- (preparing))

Let Ω be an irreducible homogeneous cone. Then one has

Ω is symmetric $\Leftrightarrow$ $\exists \underline{\nu} \neq 0$ s.t. $\Delta^{\underline{\nu}}(x)$ and $\Delta^{\underline{\nu}'}(y)$ are both non-constant polynomials.

Thus, if Ω is not symmetric, it is impossible that f and f^* both have the corresponding generalized b-functions.

$$\begin{cases} f^*(D_x)f(x)^{s+1} = b(s)f(x)^s & (x \in \Omega), \\ f(D_y)f^*(y)^{s+1} = b^*(s)f^*(y)^s & (y \in \Omega^*) \end{cases}$$

Example

Ω: Vinberg cone:

$$\Omega = \left\{ x = \begin{pmatrix} x_1 & 0 & x_2 & 0 \\ 0 & x_1 & 0 & x_4 \\ x_2 & 0 & x_3 & 0 \\ 0 & x_4 & 0 & x_5 \end{pmatrix} ;\ x_j \in \mathbb{R} \right\} \cap \mathcal{S}_4^+.$$

Assume that f has multiplier $(1,1,1)$.
Then we have $\underline{\nu} = (-1,1,1)$ and $\underline{\nu}' = (1,0,0)$.
In this case, we have $m_1 = 2$, $m_2 = 0$, $m_3 = 0$ and hence

$$b(s) = (s+2)(s+1)^2.$$

On the other hand, if f^* has multiplier $(1,1,1)$, then $\underline{\widetilde{\nu}}^* = (-1,1,1)$ and thus $\Delta^{\underline{\widetilde{\nu}}^*}(D_x)$ cannot be defined. This means that $f^*(y)$ does not have a generalized b-function.